Computation of Optimal Aircraft Trajectories Using Parameter Optimization Methods

James E. Rader*
Air Force Flight Dynamics Laboratory, Wright-Patterson Air Force Base, Ohio

and

David G. Hull†
University of Texas, Austin, Texas

The purpose of this paper is to demonstrate the applicability of parameter optimization methods to the computation of optimal aircraft trajectories. First, the optimal control problem is converted to a parameter optimization problem by chosing a form for the control which contains unknown parameters. Then, a gradient method is used to obtain approximate values for the parameters and Lagrange multipliers associated with the constraints. Finally, a second-order method is used to obtain the converged trajectory. The test problem is that of finding the angle of attack history which minimizes the time to climb of a supersonic aircraft operating at full power. The angle of attack history is approximated by a fifth-order series of Chebyshev polynomials. An inequality constraint prevents the aircraft from descending below the takeoff altitude, and this constraint is handled in both the penalty function manner and in the hard constraint manner. The latter produces a lower climb time because all the parameters can be used for optimization; also, the shape of the trajectory in the altitude-Mach number plane closely resembles that obtained from an energy-state analysis, with the transitions occurring at load factors ranging between 0.5 and 1.8.

I. Introduction

CURRENT applications of parameter optimization to the computation of optimal aerospace vehicle trajectories are with the Davidon-Fletcher-Powell method and to space-shuttle trajectories (see, for example, Ref. 1). A disadvantage of using this method is that constraints must be adjoined by penalty functions. However, to be able to adjoin constraints by Lagrange multipliers and still have good convergence characteristics requires the use of a second-order method. Hence, one purpose of this paper is to present an application of a second-order method. Another purpose is to demonstrate the applicability of parameter optimization to aircraft trajectories. To do so, the minimum time-to-climb problem is solved.

The minimum time-to-climb problem for high-performance aircraft has been the subject of many investigations. In Ref. 2, the results of two such studies are compared, that is, the results obtained from an energy-state study are compared with those obtained from a steepest-descent study. The energy-state approach uses an approximate set of equations to obtain an analytical solution. On the other hand, the steepest-descent approach uses the proper equations but arrives at an approximate result because of the convergence properties of the method. In this paper, the problem is converted to a parameter optimization problem by assuming that the angle of attack is a known function of time involving a number of unknown parameters. The values of the parameters which minimize the time to climb are then computed with a second-order method. In other words, an approximate solution is obtained because of the particular functional form chosen for the angle of attack history, but it is a completely converged solution. The degree to which the solution is approximate, however, can be reduced by making the functional form more accurate.

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*Lieutenant, U.S. Air Force. Member AIAA.

†Associate Professor, Department of Aerospace Engineering and Engineering Mechanics. Member AIAA.

II. Description of the Problem

It is desired that the angle-of-attack history be found which yields the full-throttle, minimum-time climb from a given initial state to a given final altitude and Mach number. The equations of motion are the same as Eqs. (1-6) of Ref. 2 with the additional assumptions that the angle between the thrust line and the zero-lift line is zero and that the Mach number is used as an independent variable rather than velocity. Also presented in Ref. 2 are the prescribed boundary conditions, the inequality constraint on the altitude, and the airplane aerodynamic and propulsion data (airplane 1). To obtain atmospheric properties, a constant-gravity standard atmosphere is employed.

Since a second-order optimization method is to be used, all functions must have continuous second derivatives. Hence, the aerodynamic data has been curve fit by a cubic spline; the thrust data has been approximated by the least-squares fit of a fourth-order polynominal in altitude and Mach number[3]; and the standard atmosphere has been smoothed in the region of the tropopause.

The optimal control problem discussed is converted to a parameter optimization problem by assuming that the angle of attack history is a fifth-order series of Chebyshev polynomials which contains six unknown coefficients (α_1 through α_6). With the final time being made a parameter through normalization of the time, the optimization problem involves a total of seven parameters.

To illustrate the effects of using different techniques for handling inequality constraints, the minimum-time problem has been solved two ways: a) penalty function approach and b) hard constraint approach. Each solution is discussed in Secs. III and IV.

The computations described in the paper have been carried out on a CDC 6600 using single precision arithmetic. Numerical integration is performed with a Runge-Kutta-

Fehlberg 7(8) integrator using a relative error tolerance of 10^{-6} to select the stepsize. Finally, to aid in the correction process, the altitude is computed in miles rather than feet.

III. Penalty Function Approach

In the penalty function approach, the inequality constraint $h \geq 0$ is handled by adding the following differential constraint and boundary conditions

$$P = \exp\ (-Kh)$$

$$P_i = 0, \quad P_f \equiv \text{given} \tag{1}$$

where K is a constant. The value of P_f is chosen during the first iteration for a given K, and the same value of P_f is used as K is increased. This type of penalty function has the advantage that it is on all the time regardless of whether the constant is violated or not. It also has the disadvantage that the value of $-Kh$ can exceed the maximum value allowed by the computer. If this happens, P_f must be decreased, but it must remain finite to maintain the influence of the constraint.

A. Results from the Gradient Method

To obtain values of the control parameters and the Lagrange multipliers to start the second-order method, a number of iterations are made with the gradient method proposed in Ref. 4. The gradient method only requires guesses for the control parameters; values for the multipliers are computed once the control parameters are known. Here, it is assumed that the initial angle-of-attack history is a constant 0.03 rad and that the final time is 375 sec. The latter is close to the final time estimated in Ref. 2 for the energy-state path. Finally, the constants for the penalty function are assumed to be $K=2$ and $P_f=24$; these values are used throughout the application of the gradient method.

As expected, the initial guess does not satisfy the prescribed end conditions, so the weighting constant of Ref. 4 is set equal to 0.0, and the gradient method takes corrections to satisfy the end conditions. Then, the weighting constant is set equal to 1.0 for one iteration which allows 1.0 seconds to be subtracted from the final time. After this iteration, the end conditions are not satisfied, and the previous procedure is repeated. When it is no longer possible to satisfy the end conditions to six significant figures, the value of the weighting constant is reduced to 0.5, and the iteration procedure continues. At this point, failure to satisfy the end conditions on any iteration ends the iteration process. A total of 122 iterations have been computed (74 for 1.0 and 48 for 0.5), and the results are presented in Table 1 where ν_1, ν_2, and ν_3 are the Lagrange multipliers associated with the final altitude, Mach number and penalty function, respectively.

B. Results from the Second-Order Method

The values for the final time, the control constants, and the Lagrange multipliers obtained from the gradient method are used as starting values for the second-order method discussed in Ref. 5. This method contains two weighting constants which are used to control the convergence process. The end-condition weighting constant is set equal to 1.0 to keep the end conditions satisfied, while the value of the optimization weighting constant is chosen to keep the corrections less than a certain percent of the values of the parameters. For $K=2$, the corrections are limited to 1.0% so that the optimization weighting constant starts out at 10^{-8}. A total of 47 iterations is required to achieve convergence, which is defined to occur when the norms of the control parameter corrections and the end condition dissatisfactions are less than 10^{-5}. The converged results are presented in Table 2 for increasing values of K, and the values for $K=20{,}000$ are assumed to be the desired results, that is, those which allow satisfaction of the inequality constraint. Finally, the optimal angle-of-attack history is

Table 1 Results from the gradient method ($K=2$)

Parameter	Nominal	Converged
t_f	3.75E+2	3.4500 E+2
α_1	3.00E−2	4.1327 E−2
α_2		4.4058 E−3
α_3		3.2317 E−2
α_4		−3.1226 E−3
α_5		1.2472 E−2
α_6		−6.6561 E−3
ν_1		−7.6310 E−6
ν_2		−2.3894 E−6
ν_3		3.1236 E−7

Table 2 Results from the second-order method penalty function approach

Parameter	$K=2$	$K=2000$	$K=20{,}000$
t_f	3.2379 E+2	3.2839 E+2	3.4244 E+2
α_1	4.9030 E−2	4.7561 E−2	4.6538 E−2
α_2	1.5346 E−2	1.2323 E−2	8.9312 E−3
α_3	4.0597 E−2	3.8817 E−2	3.6520 E−2
α_4	4.9345 E−3	1.5780 E−3	−2.2213 E−3
α_5	1.6255 E−2	1.5403 E−2	1.3440 E−2
α_6	−2.7160 E−3	−4.2099 E−3	−6.2389 E−3
ν_1	−3.2134 E+1	−3.7556 E+1	−8.2466 E+1
ν_2	−1.1446 E+2	−1.3083 E+2	−2.8032 E+2
ν_3	3.9788 E+0	1.0803 E−1	1.6263 E−1

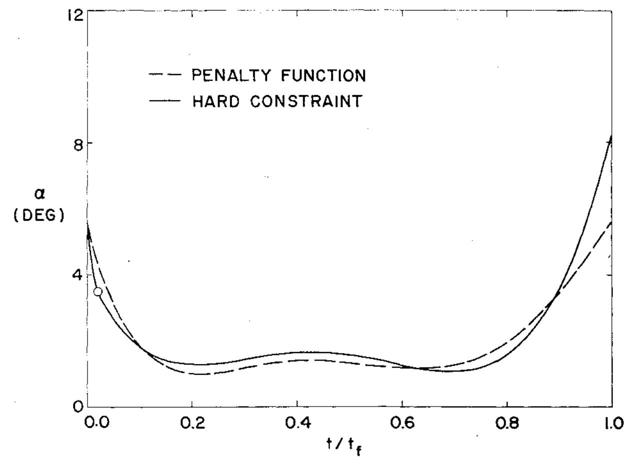

Fig. 1 Angle-of-attack histories.

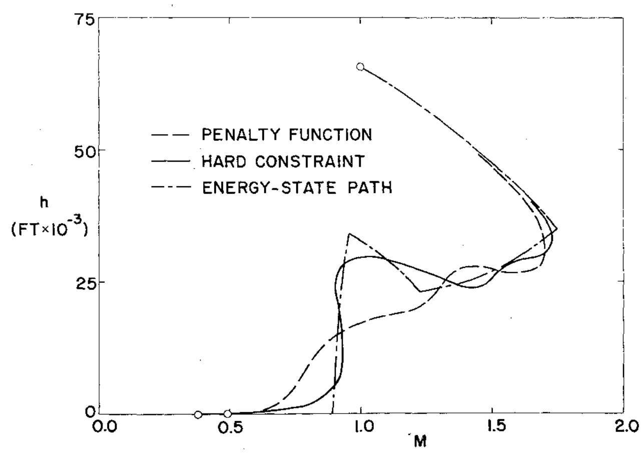

Fig. 2 Minimum time paths.

Table 3 Results from the second-order method hard constraint approach

Parameter	Nominal	Converged
t_f	3.4244 E+2	3.1748 E+2
α_1	4.6538 E−2	5.0689 E−2
α_2	8.9312 E−3	2.3821 E−2
α_3	3.6520 E−2	4.0615 E−2
α_4	−2.2213 E−3	1.2786 E−2
α_5	1.3440 E−2	1.7260 E−2
α_6	−6.2389 E−3	−8.9000 E−4
ν_1	−8.2466 E+1	−2.9548 E+1
ν_2	−2.8032 E+2	−1.0807 E+2

shown in Fig. 1, and the shape of the trajectory in the altitude-Mach number plane is presented in Fig. 2.

IV. Hard Constraint Approach

In the hard constraint appraoch, the inequality constraint is satisfied exactly rather than in an approximate manner as it is in the penalty function approach. When the optimal control causes the aircraft to violate the inequality constraint, it is replaced by the control which causes the aircraft to fly the constraint. The points at which the aircraft goes on and off the boundary must be determined with precision—to at least to the accuracy that the differential equations are being integrated.

For the problem under consideration, the results of Sec. III show that the inequality constraint is in effect at the beginning of the trajectory. Hence, the control history has been obtained by using the angle of attack for $h=0$ until it is equal to the optimal control and then switching to the optimal control. The results obtained in Sec. IIIB for $K=20{,}000$ are used to start the second-order method, and the values shown in Table 3 are obtained after 17 iterations.‡ Note the difference in the minimum time t_f for the hard constraint approach and the penalty function approach. The reason for this is that the polynominal control can optimize more effectively if it does not have to satisfy the inequality constraint.

The optimal angle-of-attack history is presented in Fig. 1, while the shape of the trajectory in the altitude-Mach number plane is shown in Fig. 2. For the sake of comparison, the energy-state path is shown in Fig. 2, and it is interesting to note the similarity of the two. Other characteristics of the optimal path which are of interest include the normal acceleration and path inclination histories. The following are extreme values of the normal acceleration encountered during the climb: 0.5 during the pull-up to begin the subsonic climb, −0.5 and 0.2 during the transition from the subsonic climb to the supersonic climb, and 0.8 during the transition to the zoom climb at the end of the trajectory. If it is assumed that the normal acceleration is approximately equal to the load factor minus one, the load factor varies between 0.5 and 1.8 during the optimal climb. With regard to the path inclination, the following extreme values have been computed: 30° during the subsonic climb, −6° during the transition from the subsonic climb to the supersonic climb, and 41° at the end of the zoom climb. While these results are interesting, it should be kept in mind that they have been obtained from a fifth-order polynomial approximation for the angle of attack and that the results could change by increasing the order of the polynomial.

‡There is no value for ν_3 because the constraint (1) is no longer used.

V. Conclusions

It has been shown that optimal aircraft trajectories can be computed effectively using parameter optimization methods, particularly second-order methods. If inequality constraints are present, they can be handled in either a penalty function manner or a hard constraint manner. However, for the former, a higher-order polynomial control is needed to get close to the optimum. It should be mentioned that the approach followed to obtain the hard-constraint solution was composed of three steps: 1) penalty function, gradient method, 2) penalty function, second-order method, and 3) hard constraint, second-order method. No attempt has been made to obtain the hard-constraint solution directly from the constant angle of attack guess.

With regard to the problem solved, that is, the minimum time-to-climb problem it is shown that the optimal trajectory in the altitude-Mach number plane is quite close to the energy-state path. Furthermore, the load factor for the optimal path varies in the range 0.5 to 1.8.

There are several advantages to using parameter optimization methods for solving optimal control problems 1) The method is easy to program. 2) A very general program can be written which would require little modification for solving new problems. 3) First and second derivatives can be computed numerically; hence, only the equations of motion have to be integrated. 4) Since the control is actually being guessed, it is not difficult to obtain reasonable guesses for the parameters. 5) The minimal or maximal nature of a control history can easily be verified by showing that the second variation is positive; this amounts to proving a matrix is positive definite.

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